If the function

$$\lambda(\zeta) = \left(\frac{p}{\pi}\right)^2 (\omega^2\mu_0\epsilon - \gamma^2) \quad (6)$$

is expansible in an absolutely convergent Fourier cosine series [5]

$$\lambda(\zeta) = \theta_0^{(h)} + 2\sum_{n=1}^{\infty} \theta_n^{(h)} \cos 2n\zeta \quad (7)$$

(5) becomes

$$\frac{d^2f^{(h)}}{d\zeta^2} + \left(\theta_0^{(h)} + 2\sum_{n=1}^{\infty} \theta_n^{(h)} \cos 2n\zeta\right) f^{(h)} = 0 \quad (8)$$

which is the canonical form of Hill's equation. The computation of stability charts for Hill's equation, and, thus, the determination of the pass band and stop band structure of the dispersion characteristics, follows from the characteristic equation for Hill's equation

$$\sin^2 \frac{\pi\beta}{2} = \Delta^{(h)}(0) \sin^2 \frac{\pi\sqrt{\theta_0^{(h)}}}{2} \quad (9)$$

where β denotes the propagation factor in the Floquet solution to Hill's equation and $\Delta^{(h)}(0)$ is an infinite determinant whose elements are

$$\Delta^{(h)}(0)|_{mm} = 1 \quad (10)$$

$$\Delta^{(h)}(0)|_{mn} = \frac{\theta^{(h)}m - n}{\theta_0^{(h)} - 4m^2} \; (m \neq n). \quad (11)$$

This procedure has been described in [1] and [3]. The solution to Hill's equation may be obtained, when β is known, by means of a procedure outlined in [5] and [6]

In the case of TM wave propagation, one introduces into (2) the substitutions

$$\zeta = \frac{\pi z}{p} \quad (12)$$

$$U^{(e)}(z) = \epsilon^{1/2} f^{(e)}(\zeta) \quad (13)$$

yielding the differential equation for $f^{(e)}(\zeta)$,

$$\frac{d^2f^{(e)}}{d\zeta^2} + \left[\frac{1}{2\epsilon}\frac{d^2\epsilon}{d\zeta^2} - \frac{3}{4\epsilon^2}\left(\frac{d\epsilon}{d\zeta}\right)^2 + \left(\frac{p}{\pi}\right)^2 (\omega^2\mu_0\epsilon - \gamma^2)\right] f^{(e)} = 0. \quad (14)$$

If ϵ is an even-periodic function, so also is the function in square brackets and, thus, one may write

$$[\quad] = \theta_0^{(e)} + 2\sum_{n=1}^{\infty} \theta_n^{(e)} \cos 2n\zeta \quad (15)$$

if the series is absolutely convergent. Hence,

$$\frac{d^2f^{(e)}}{d\zeta^2} + \left[\theta_0^{(e)} + 2\sum_{n=1}^{\infty} \theta_n^{(e)} \cos 2n\zeta\right] f^{(e)} = 0 \quad (16)$$

which is again the canonical form of Hill's equation.

Thus, the z dependence of both TE and TM waves in periodic media is expressible in terms of Hill functions. The pass band and stop band characteristics or ω-β diagrams may be determined from the characteristic equation by numerical or graphical methods and the functional dependence of the fields from the solutions to the Hill equation.

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Note on the Measurement of Material Properties by the Strip-Line Cavity

It has been found that when making measurements of the properties of materials with a strip-line cavity [1]–[3], results are obtained which are consistently lower than expected. The error is the more serious, the higher the value of the dielectric constant or permeability, as the case may be, of the sample.

In the case of measurements of magnetic properties, the reason for the effect has been explained elsewhere [4]. The discrepancy is attributable to demagnetizing factors in the specimen, and when this is in the form of a flat slab, placed either vertically against the end wall of the cavity or horizontally on the strip, the true relative permeability of an isotropic specimen $\bar{\mu}$ is given by

$$\bar{\mu} = \frac{\mu(1 - N)}{1 - \mu N} \quad (1)$$

where μ is the apparent permeability given by the perturbation formulae of [1] and [3], and N is the demagnetizing factor of the specimen appropriate to the direction of the microwave magnetic field (in MKS units). It is apparent from this formula that the difference between μ and $\bar{\mu}$ increases with increasing μ and $\bar{\mu}$, being zero for $\mu=\bar{\mu}=1$. For large values of $\bar{\mu}$, μ approaches the limiting value $1/N$.

In the case of measurements of the dielectric constant, the discrepancy is attributable to the presence of minute air gaps between the specimen and the strip and ground plane. The perturbation formula of [1] and [3] gives a value for the dielectric constant ϵ which would be correct if the sample fitted flush with the strip and ground plane. If there is an appreciable gap, the value of ϵ calculated from the perturbation formula is an apparent one. The relation between ϵ and the true dielectric constant $\bar{\epsilon}$ can be calculated by means of the concept of a dielectric circuit analogous to the well-known magnetic circuit. The result is

$$\bar{\epsilon} = \frac{\epsilon(1 - x/h)}{1 - \epsilon x/h} \quad (2)$$

where x is the total gap height, i.e., the sum of the gaps at top and bottom of the sample, and h is the distance from the strip to the ground plane, i.e., the distance $b-t$ in the notation of [1]–[3]. It is apparent from this formula that the difference between ϵ and $\bar{\epsilon}$ increases with increasing ϵ and $\bar{\epsilon}$, being zero for $\epsilon=\bar{\epsilon}=1$. For large values of $\bar{\epsilon}$, ϵ approaches the limiting value h/x.

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On Mode Losses in Confocal Resonator and Transmission Systems

In a recent correspondence Lonngren and Beyer [1] calculated the losses for a single iteration in a "beam waveguide" [2] with circular lenses separated by twice their focal length. Since the confocal Fabry-Perot resonator with two identical circular mirrors may be studied by superimposing two guided wave beams propagating oppositely in the given system, the beam-waveguide losses allow one to determine the resonator Q. The problem which Lonngren and Beyer [1] have solved approximately is to find the eigenvalues $\gamma_{\alpha,n}(c)$ of the integral equation

$$\gamma_{\alpha,n}(c)\mathcal{S}_{\alpha,n}(c, x) = \int_0^1 cJ_\alpha(cxy)\mathcal{S}_{\alpha,n}(c, y)y\,dy \quad (1)$$

for small c. In an earlier work Beyer and Scheibe [3] obtained values of $\gamma_{\alpha,n}(c)$ for large c. The purpose of this correspondence is to point out that the same information has been obtained by directly studying the solutions of (1).

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